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Quantum chaos, random matrix theory, statistical mechanics in two dimensions, and the second law—a case study

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Abstract. We present a theory where the statistical mechanics for dilute ideal gases can be derived from the random matrix approach. We show the connection of this approach with the Srednicki approach which connects Berry conjecture with statistical mechanics. We further establish a link between Berry conjecture and random matrix theory. In the course of arguing for these connections, we also observe sum rules associated with the outstanding counting problem in the theory of Braid groups. We believe that these arguments, developed for a special example connecting the properties of eigenfunctions and random matrices to the second law of thermodynamics, will eventually prove to be more general.

1. Introduction

We begin with a brief overview of various different links that have been discovered in the last few decades between classically chaotic systems and their quantal counterparts. Any study motivated to bring about this connection is what we understand here by 'quantum chaos' [1, 2]. An overwhelming number of numerical experiments on spectral statistics [3, 4] and their corresponding semiclassical analysis [5, 6] suggest that the universal features observed in chaotic quantum systems can be modelled in terms of random matrix theory (RMT). Apart from energy spectra, it has been found that the conjecture [7] where an eigenstate of a chaotic quantum system is represented as a Gaussian random superposition of plane waves entails results which are found in agreement with numerical studies [8, 4]. We believe that an important step has been in establishing the result that this conjecture leads to momentum distribution of ideal gases, thus bringing out statistical mechanics [9]. However, in order to bring out the puzzling results in two-dimensional statistical mechanics, it is necessary that the choice of the correlations between amplitudes of the eigenstates is specified. Thus, in this pursuit, we are led to RMT where one can systematically choose the ensemble. Recently, it has been shown how one can go from RMT to statistical mechanics [10]—a work that has brought together two important statistical theories which have been, hitherto, considered quite apart.

Throughout this paper, we will be concentrating on two dimensions as that is the most difficult case in statistical mechanics [11–13]. In section 2, we give a brief discussion of the choice of random matrix ensemble when time-reversal and parity are broken. This is fundamental in dealing successfully with the problem of momentum distribution function and virial coefficients in section 3. The fact that quantum mechanics can be done on real field if the antiunitary symmetries are well specified [15], and, the classification theorem of associative division algebra [17] leads to three basic ensembles in RMT [16]. Incorporating the violation of parity is an important step. In section 4, we unify the different streams of thought from quantum chaos, RMT, and statistical mechanics by discussing entropy which is fundamental to all the three. We would like to mention that a recent work [18] is an interesting companion of this paper. We conclude the paper with a summary.

Before we begin, let us mention that our discussion here is restricted (in the present form) to random matrix ensembles that are invariant under a canonical group. Indeed, it is not necessary at all that various statistical properties of the spectrum and eigenfunctions of chaotic quantum systems follow the canonical RMT. It is known that, upon taking localization effects into account, the eigenfunction statistics is considerably modified [19]. In a given system of many identical particles, it may become necessary to consider in the ansatz for eigen- (or wave)-function these non-universal features.

2. Random matrix ensemble in two dimensions

Usually in discussions on RMT, the space dimensionality of the physical system plays no explicit role. Due to the subtleties arising from the fact that we are working in two dimensions, we present here a comparative discussion about the fundamental symmetries in two and three (or greater) space dimensions which decisively restrict the possibilities of the random matrix ensemble. Let us emphasize that this does not mean that our discussion becomes incompatible with the observation that spectral statistics and eigenfunction statistics is found in agreement with RMT. The discussion below is only meant for a many-body system of identical particles.

Denoting the time reversal operator by $\hat{T}$, the position operator by $\hat{q}$, the momentum operator by $\hat{p}$, and the spin angular momentum by $\hat{\sigma}$ satisfy

$$\begin{aligned} \hat{T}\hat{q}\hat{T}^{-1} &= \hat{q} \\ \hat{T}\hat{p}\hat{T}^{-1} &= -\hat{p} \\ \hat{T}\hat{\sigma}\hat{T}^{-1} &= -\hat{\sigma}. \end{aligned} \tag{1}$$

In order to preserve the commutator between $\hat{q}$ and $\hat{p}$, we see through

$$\hat{T}\mathrm{i}\hat{T}^{-1} = -\mathrm{i} \tag{2}$$

(i is the square root of -1) that $\hat{T}$ is antilinear. Moreover, since

$$\hat{T}\hat{T}^{\dagger} = 1 \tag{3}$$

we say that $\hat{T}$ is antiunitary. This then leads to

$$\hat{T}^2 = \pm 1 \tag{4}$$

giving us the possibilities—symmetric and antisymmetric states of a physical system which are consistent with even and half-odd integral spin respectively yielding the Bose–Einstein and Fermi–Dirac distributions. On very general grounds thus, if a system respects time reversal symmetry, the Hamiltonian can be represented in terms of real or quaternion real elements depending on spin and rotational symmetry. If, however, time reversal is broken, the elements are complex, and the canonical group that preserves the Hamiltonian is unitary. Therefore, in three (or greater) space dimensions, a random matrix ensemble can be chosen appropriately satisfying the invariance under an orthogonal, a unitary or a symplectic group, and no more [17].

It is important to note now that the fact that we have only symmetric or antisymmetric states here shows that the space dimensions must be three or greater since in two dimensions, there is an extra phase factor under an exchange of two coordinates which leads to a fractional angular momentum leading to fractional statistics. In the gauge where the two particles (in case of a discussion of two particles one can just consider the centre of mass as one particle) are free, the boundary conditions get twisted. For a free charged particle of charge Q in a magnetic flux, Φ, the boundary condition is $\Psi(\varphi) \sim \exp[\mathrm{i}(\text{integer} - Q\Phi/2\pi)\varphi]$; we see that the angular momentum becomes fractional [21, 13]. This leads to a distinction between clockwise and anticlockwise rotations, which leads us to the notion of chirality and the associated breakdown of parity. In general, in two space dimensions, parity and time reversal symmetries are broken. Any choice of a random matrix ensemble must be consistent with this.

Since time reversal is broken, it follows from the foregoing discussion that the Hamiltonian matrix of the system will be complex, invariant under a unitary group. Breakdown of parity is new, however, the answer is in the boundary condition. Thus, we are led to a chiral unitary ensemble.

In our present context of a many-body system whose eigenstates we cannot know exactly due to practical limitations (even if it is possible in some cases, we deal with the situation where a statistical study is the viable option), following Srednicki, we write a random pure state as a superposition of some basis states with amplitudes which are random. By the randomness of the amplitudes, we mean that they satisfy some correlation functions which we will write in the next section. The randomness in the amplitudes makes the pure states of the system also random. We have assumed that the system is isolated.

The randomness in the pure state can also be interpreted [22] by weighting the eigenvectors by a measure invariant under unitary transformations, U_N. By considering the unit complex N-sphere as a homogeneous space of U_N, then again as U_N itself but organized into cosets, the measure is seen to be the Haar measure on U_N, thus unique. It is from this interpretation that we will discuss the entropy of the subsystem where we will note a connection between randomness in pure state and RMT, however, in that case it will be applied to the density operator of the subsystem which resides in the isolated system with a Hilbert space of lesser dimensionality.

3. Momentum distribution

Let us consider a system of N hard spheres (‘discs’ in two dimensions), each of radius a, enclosed in a box of edge length $L + 2a$. Centres of two hard spheres $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are such that $|\boldsymbol{x}_i - \boldsymbol{x}_j| \geqslant 2a$. The canonical pair of coordinates describing these particles are $(\boldsymbol{X}, \boldsymbol{P})$ where $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N), \boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N)$. Energy eigenfunctions, $\psi_\alpha(\boldsymbol{X})$ corresponding to eigenvalue E_α vanish on the boundary of the enclosure. A typical eigenfunction is irregular, with a Gaussian amplitude distribution and the spatial correlation function of the same is consistent with the conjecture of Berry which allows us to represent this eigenfunction as a superposition, following Srednicki [9]:

$$\psi_\alpha(\boldsymbol{X}) = N_\alpha \int \mathrm{d}^{dN}\boldsymbol{P}\, A_\alpha(\boldsymbol{P})\delta(P^2 - 2mE_\alpha)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\boldsymbol{X}\cdot\boldsymbol{P}} \tag{5}$$

with N_α given by the normalization constant, and A_α's satisfying the two-point correlation function

$$\langle A_\alpha^*(\boldsymbol{P}) A_\gamma(\boldsymbol{P}')\rangle_{\mathrm{ME}} = \delta_{\alpha\gamma} \frac{\delta^{dN}(\boldsymbol{P}-\boldsymbol{P}')}{\delta(\boldsymbol{P}^2-\boldsymbol{P}'^2)} \tag{6}$$

where d denotes the number of coordinate-space dimensions. The average in (6) is a matrix-ensemble (ME) average which originates from the fact that the Hamiltonian, H, of the system belongs to an ensemble of matrices satisfying associative division algebra [16, 17] in consistency with quantum mechanics. The eigenstate ensemble (EE) used in [9] is nothing but a consequence of underlying matrix ensemble in RMT, the eigenfunctions then satisfy all the properties numerically observed and analytically represented in (5), (6) and [20]. The correlation functions (6) decide whether time-reversal symmetry is preserved ($A_\alpha^*(\boldsymbol{P}) = A_\alpha(-\boldsymbol{P})$) or broken ($A_\alpha^*(\boldsymbol{P}) \neq A_\alpha(-\boldsymbol{P})$), accordingly the corresponding matrix ensemble belongs to orthogonal ensemble (OE) or unitary ensemble (UE) respectively. As noted in [9], the higher-order even-point correlation functions factorize and the odd ones vanish. A very important aspect of the ansatz (5), (6) is that the Wigner function corresponding to $\psi_\alpha(\boldsymbol{X})$ is microcanonical, or, is proportional to $\delta(H - E_\alpha)$ which, in a sense, incorporates ergodicity. We note here that, starting from an ansatz very similar to those above, it is possible to obtain the quantum transport equation [23] where it is important to relate a given quantum state with the admissible energy surface in phase space; thus the above ansatz is in conceptual agreement with the ergodic aspect of a many-body system. Moreover, this choice fixes the Thomas–Fermi density of states naturally. It now becomes important to emphasize that we must restrict ourselves to dilute gas of hard spheres and also assume that the size of sphere is much less than the thermal de Broglie wavelength. Thus, the ansatz establishes, in fact, a link between RMT and statistical mechanics. We now incorporate the case of two dimensions which otherwise presents enormous difficulties.

In two dimensions, the solutions of the Schrödinger equation, $\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$, under an exchange of two coordinates of particles satisfies

$$\psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_N) = \mathrm{e}^{\mathrm{i}\pi\nu} \psi(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \ldots, \boldsymbol{x}_i, \ldots, \boldsymbol{x}_N) \tag{7}$$

where ν is arbitrary and defines statistics. For $\nu = 0$ and $\nu = 1$, with (6), one gets the Bose–Einstein and Fermi–Dirac distributions. This non-trivial phase and the resulting boundary condition arises from the fact that the effective configuration space, M_N^2 has a fundamental group, $\pi_1(M_N^2) = B_N$ [24], the Braid group of N objects which is an infinite, non-Abelian group. B_N is generated by $(N-1)$ elementary moves $\sigma_1, \ldots, \sigma_{N-1}$ satisfying the Artin relations,

$$\begin{aligned} &\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad (i = 1, 2, \ldots, N-2) \\ &\sigma_j \sigma_i = \sigma_i \sigma_j \qquad |i - j| \geqslant 2 \end{aligned} \tag{8}$$

the inverse of σ_i is σ_i^{-1}, the identity is denoted by I, and the centre of B_n is generated by $(\sigma_1\sigma_2 \cdots \sigma_{N-1})^N$. The multivaluedness of the eigenfunction originates from the phase change in effecting an interchange between two coordinates $x_i^{(1)}$ and $x_i^{(2)}$ (where superscripts refer to components) which can be expressed as

$$\begin{aligned} V &= \exp\left(\mathrm{i}\nu \sum_{i<j} \phi_{ij}\right) \\ \phi_{ij} &= \tan^{-1}\left(\frac{x_i^{(2)} - x_j^{(2)}}{x_i^{(1)} - x_j^{(1)}}\right). \end{aligned} \tag{9}$$

The description adopted by us here is referred to as the anyon gauge. It is important to realize that a set of coordinate configuration can be reached starting from some initial coordinates of N particles in an infinite way, each possibility manifested by an action of an element $\beta \in B_N$.

The connection between initial and final sequences is given by (7), via the character $\chi(\beta)$ of the specific element. Thus, to every $\beta \in B_N$, we can associate the affected partial amplitude $\psi_\alpha(\beta : \boldsymbol{x})$ [25]. With one-dimensional unitary representation of the braid group, the rudiments of quantum mechanics allow us to write

$$\Phi_\alpha(\boldsymbol{X}) = \sum_{\beta \in B_n} \chi(\beta)\psi_\alpha(\beta : \boldsymbol{X}) \tag{10}$$

where $\psi_\alpha(\beta : \boldsymbol{X})$ is the probability amplitude associated in changing a configuration $\boldsymbol{X}$ to $(\beta : \boldsymbol{X})$—a configuration after the action of β on $\boldsymbol{X}$. The wavefunction $\Phi_\alpha(\boldsymbol{X})$ is to be understood as appropriately normalized. The ansatz for $V\psi_\alpha(\beta : \boldsymbol{X})$ is now

$$V\psi_\alpha(\beta : \boldsymbol{X}) = N_\alpha \int \mathrm{d}^{2N}\boldsymbol{P}\, A_\alpha(\beta : \boldsymbol{P})\delta(P^2 - 2mE_\alpha)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\boldsymbol{X}\cdot\boldsymbol{P}} \tag{11}$$

with $A_\alpha(\beta : \boldsymbol{P})$ satisfying

$$\langle A^*_\alpha(\beta_1 : \boldsymbol{P}_1)A_\gamma(\beta_2 : \boldsymbol{P}_2)\rangle_{\mathrm{ME}} = \delta_{\alpha\gamma}\frac{\delta^{2N}((\beta_1 : \boldsymbol{P}_1) - (\beta_2 : \boldsymbol{P}_2))}{\delta(\boldsymbol{P}_1^2 - \boldsymbol{P}_2^2)} \tag{12}$$

$(\beta_1, \beta_2 \in B_N)$, and $A_\alpha(\boldsymbol{P})$ satisfy the twisted boundary conditions,

$$A_\alpha(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_i, \ldots, \boldsymbol{p}_j, \ldots, \boldsymbol{p}_N) = \mathrm{e}^{\mathrm{i}\pi\nu}A_\alpha(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_j, \ldots, \boldsymbol{p}_i, \ldots, \boldsymbol{p}_N). \tag{13}$$

The question now is to specify exactly what the matrix ensemble is in this case? The form of (12) with A_α's not restricted to real, takes into account the T-breaking, and (13) makes the ensemble handed or chiral as a result of P-breaking. Thus (11)–(13) gives the complete description and the ME is, in fact, the chiral-Gaussian unitary ensemble (ch-GUE) [26] as discussed in the previous section from general considerations. It can easily be shown that the Wigner distribution is

$$\begin{aligned} \langle \rho^{\mathrm{W}}_\alpha(\boldsymbol{X}, \boldsymbol{P})\rangle_{\mathrm{ME}} &= n_\alpha^{-1}h^{-2N}\delta\left(\frac{P^2}{2m} - E_\alpha\right) \\ n_\alpha &= \frac{1}{N!\Gamma(N)E_\alpha}\left(\frac{mL^2E_\alpha}{2\pi\hbar^2}\right)^N. \end{aligned} \tag{14}$$

For the momentum distribution, we need to evaluate the ME average of $\tilde{\Phi}^*_\alpha(\boldsymbol{P})\tilde{\Phi}^*_\gamma(\boldsymbol{P}'')$ with $\tilde{\Phi} \equiv V\Phi_\alpha$. With the above ansatz and conditions supplementing it, this average is

$$\begin{aligned} \mathcal{F}(\boldsymbol{P}) = \langle \tilde{\Phi}^*_{\alpha'}(\boldsymbol{P})\tilde{\Phi}^*_\gamma(\boldsymbol{P}'')\rangle_{\mathrm{ME}} &= h^{2N}\delta_{\alpha'\gamma}N_{\alpha'}N_\gamma \sum_{n,m=0}\sum_{\beta_1(m)}\sum_{\beta_2(n)}\chi^*(\beta_1)\chi(\beta_2)\delta(P^2 - 2mE_{\alpha'}) \\ &\times \delta^{2N}_{\mathcal{D}}\left(\prod_{\alpha=0}^{m}\sigma^{\epsilon_{\beta_1}}_{\beta_1(\alpha)}\boldsymbol{P}'' - \prod_{\alpha=0}^{n}\sigma^{\epsilon_{\beta_2}}_{\beta_2(\alpha)}\boldsymbol{P}\right)\Bigg|_{\boldsymbol{P}=\boldsymbol{P}''} \end{aligned} \tag{15}$$

where

$$\delta^{2N}_{\mathcal{D}}(\boldsymbol{Q}) = h^{-2N}\int_{\mathrm{Domain},\mathcal{D}} \mathrm{d}^{2N}X\, \exp\left(\frac{\mathrm{i}}{\hbar}\boldsymbol{Q}\cdot\boldsymbol{X}\right) \tag{16}$$

$\boldsymbol{P}$ is identified with $\boldsymbol{P}''$ after the sum is performed.

With (15), the momentum distribution is given by

$$F(\boldsymbol{p}_1) = \frac{\int \mathrm{d}\boldsymbol{p}_2 \ldots d\boldsymbol{p}_N\mathcal{F}(\boldsymbol{P})}{\int \mathrm{d}\boldsymbol{p}_1 \ldots d\boldsymbol{p}_N\mathcal{F}(\boldsymbol{P})} \tag{17}$$

which formally completes the deduction. However, an exact evaluation of this is very difficult and the difficulty is due to counting the irreducible words formed by the σ's. To make the precise connection, we derive the result up to $\mathrm{O}(\hbar^2/L^2)$, an order that is enough for second virial coefficient.

In deriving the momentum distribution, we have to consider all the exchanges that lead to contributions giving second virial coefficient. With N generators, we have characters $\mathrm{e}^{\mathrm{i}\pi\nu}$ and $\mathrm{e}^{-\mathrm{i}\pi\nu}$ leading to a combination, $\cos(N\pi\nu)$. With one of the momenta fixed in the above integral, we look for elements of B_N such that two momenta are interchanged restoring all other momenta to their labellings. All these elements contribute up to $\mathrm{O}(\hbar^2/L^2)$. There are two kinds of terms with any σ_M (which denotes the elements of B_N with M generators):

(i) one where $\boldsymbol{p}_1$ changes;

(ii) one where $\boldsymbol{p}_1$ does not change.

We first insert a notation which will be used in sequel, namely, the integral,

$$\int \mathrm{d}^N p\, \delta(p^2 - x) := I_N(x) = \frac{(\pi x)^{N/2}}{\Gamma(N/2)x}. \tag{18}$$

In case (ii), we have typically

$$\delta_D(\boldsymbol{p}_1 - \boldsymbol{p}_1)\delta_D(\boldsymbol{p}_2 - \boldsymbol{p}_2)\ldots\delta_D(\boldsymbol{p}_j - \boldsymbol{p}_i)\delta_D(\boldsymbol{p}_i - \boldsymbol{p}_j)\ldots\delta_D(\boldsymbol{p}_N - \boldsymbol{p}_N). \tag{19}$$

This leads to the value of the integral,

$$\frac{1}{2} I_{2(N-2)}(2mE_\alpha - p_1^2)\left(\frac{L}{h}\right)^{2(N-1)} \mathcal{R}_{\boldsymbol{\sigma}_M}(N)\chi(\boldsymbol{\sigma}_M) \tag{20}$$

where $\mathcal{R}_{\boldsymbol{\sigma}_M}$ denotes the number of elements composed by M generators that contribute to $\mathrm{O}(h^2/L^2)$, or just an interchange between two momenta but not $\boldsymbol{p}_1$, and, $\chi(\boldsymbol{\sigma}_M)$ denotes the corresponding character.

In case (i), we have typically

$$\delta_D(\boldsymbol{p}_2 - \boldsymbol{p}_1)\delta_D(\boldsymbol{p}_1 - \boldsymbol{p}_2)\ldots\delta_D(\boldsymbol{p}_N - \boldsymbol{p}_N) \tag{21}$$

which leads to the integral evaluating to

$$I_{2(N-2)}(2mE_\alpha - 2p_1^2)\left(\frac{L}{h}\right)^{2(N-1)} \mathcal{Q}_{\boldsymbol{\sigma}_M}(N)\chi(\boldsymbol{\sigma}_M) \tag{22}$$

where $\mathcal{Q}_{\boldsymbol{\sigma}_M}(N)$ denotes the number of elements of B_N composed by N generators contributing to $\mathrm{O}(h^2/L^2)$ that involve an interchange with $\boldsymbol{p}_1$.

For each $\boldsymbol{\sigma}_M \to \chi(\boldsymbol{\sigma}_M)$, we can find $\boldsymbol{\sigma}_M^* \to \chi^*(\boldsymbol{\sigma}_M)$, and $(\mathcal{Q}_{\boldsymbol{\sigma}_M}, \mathcal{R}_{\boldsymbol{\sigma}_M}) = (\mathcal{Q}_{\boldsymbol{\sigma}_M^*}, \mathcal{R}_{\boldsymbol{\sigma}_M^*})$. Thus, for fixed M, case (i) gives

$$\left(\frac{L}{h}\right)^{2(N-1)} I_{2(N-2)}(2mE_\alpha - 2p_1^2)\mathcal{Q}_{\boldsymbol{\sigma}_M}(N)(\chi(\boldsymbol{\sigma}_M) + \chi^*(\boldsymbol{\sigma}_M)) \tag{23}$$

and case (ii) gives

$$\left(\frac{L}{h}\right)^{2(N-1)} \frac{1}{2} I_{2(N-2)}(2mE_\alpha - p_1^2)\mathcal{R}_{\boldsymbol{\sigma}_M}(N)(\chi(\boldsymbol{\sigma}_M) + \chi^*(\boldsymbol{\sigma}_M)). \tag{24}$$

Because only two momenta are interchanged, the total contribution of elements of B_N formed by M generators is

$$\left(\frac{L}{h}\right)^{2(N-1)} \sum_{k=-M,-M+2,\ldots,M-2,M} [I_{2(N-2)}(2mE_\alpha - p_1^2)\cos(\pi k\nu)\mathcal{R}_k^{(M)}(N) \\ + 2I_{2(N-2)}(2mE_\alpha - 2p_1^2)\cos(\pi k\nu)\mathcal{Q}_k^{(M)}(N)] \tag{25}$$

for

$$\chi(\boldsymbol{\sigma}_M) = [\mathrm{e}^{-\mathrm{i}M\pi\nu}, \mathrm{e}^{-\mathrm{i}(M-2)\pi\nu}, \ldots, \mathrm{e}^{\mathrm{i}M\pi\nu}] = \mathrm{e}^{\mathrm{i}k\pi\nu} \tag{26}$$

and, $\mathcal{R}_{\boldsymbol{\sigma}_M}$ is just $\mathcal{R}_k^{(M)}(N)$. If we integrate over $\boldsymbol{p}_1$, we obtain the normalization factor. For this, we have terms that lead to an exchange as discussed above, and also the operation of identity of B_N where no momenta are changed. To begin with, we have the integration of the term without identity, and the result is

$$\left(\frac{L}{h}\right)^{2(N-1)} \sum_{k=-M,-M+2,\ldots,M-2,M} [I_{2(N-1)}(2mE_\alpha)\cos(\pi k\nu)\mathcal{R}_k^{(M)}(N) + 2I_{2(N-1)}(2mE_\alpha)\cos(\pi k\nu)\mathcal{Q}_k^{(M)}(N)]. \tag{27}$$

Denoting by $\mathcal{P}_{\boldsymbol{\sigma}_M}(N)$ the number of elements of B_N composed of M generators contributing to $\mathrm{O}(h^0/L^0)$—the identity, integration over $\boldsymbol{p}_2 \ldots \boldsymbol{p}_N$ gives

$$\left(\frac{L}{h}\right)^{2N} I_{2(N-1)}(2mE_\alpha - p_1^2)\mathcal{P}_{\boldsymbol{\sigma}_M}(N)\chi(\boldsymbol{\sigma}_M). \tag{28}$$

As above, we have $\boldsymbol{\sigma}_M$ and $\boldsymbol{\sigma}_M^*$, so this integral reduces to

$$\left(\frac{L}{h}\right)^{2N} \sum_{k=-M,-M+2,\ldots,M-2,M} 2I_{2(N-1)}(2mE_\alpha - p_1^2)\cos(\pi k\nu)\mathcal{P}_k^M(N). \tag{29}$$

To get the contribution of identity to normalization, we now integrate this over $\boldsymbol{p}_1$ to obtain

$$\left(\frac{L}{h}\right)^{2N} \sum_{k=-M,-M+2,\ldots,M-2,M} 2I_{2N}(2mE_\alpha)\cos(\pi k\nu)\mathcal{P}_k^M(N). \tag{30}$$

For the second virial coefficient, if $M = 2m(m = 0, 1, 2, \ldots)$, the contribution goes to the term involved in the identity, and, if $M = 2m + 1(m = 0, 1, 2, \ldots)$, the contribution is $\mathrm{O}(h^2/L^2)$. The sum over elements of B_N can be substituted by a sum over m. All put together, in the term which gives normalization, we have

$$\mathrm{O}(1) : \left(\frac{L}{h}\right)^{2N} \sum_{m=0}^{\infty} \sum_{k=-2m,-2m+2,\ldots 2m} 2I_{2N}(2mE_\alpha)\cos(\pi k\nu)\mathcal{P}_k^M(N) \tag{31}$$

and

$$\mathrm{O}\left(\frac{h^2}{L^2}\right) : \left(\frac{L}{h}\right)^{2(N-1)} \sum_{m=0}^{\infty} \sum_{k=-2m-1,-2m+1,\ldots 2m+1} [I_{2(N-1)}(2mE_\alpha)\cos(\pi k\nu)\mathcal{R}_k^M(N) + I_{2(N-1)}(2mE_\alpha)\cos(\pi k\nu)\mathcal{Q}_k^M(N)]. \tag{32}$$

For the numerator of (19), with one momentum $\boldsymbol{p}_1$ fixed and integrating with respect to all other momenta, we get the following results:

$$\mathrm{O}(1) : \left(\frac{L}{h}\right)^{2N} \sum_{m=0}^{\infty} \sum_{k=-2m,-2m+2,\ldots 2m} 2I_{2(N-1)}(2mE_\alpha - p_1^2)\cos(\pi k\nu)\mathcal{P}_k^M(N) \tag{33}$$

and

$$\mathrm{O}\left(\frac{h^2}{L^2}\right) : \left(\frac{L}{h}\right)^{2(N-1)} \sum_{m=0}^{\infty} \sum_{k=-2m-1,-2m+1,\ldots 2m+1} [I_{2(N-2)}(2mE_\alpha - p_1^2)\cos(\pi k\nu)\mathcal{R}_k^M(N) + 2I_{2(N-2)}(2mE_\alpha - 2p_1^2)\cos(\pi k\nu)\mathcal{Q}_k^M(N)]. \tag{34}$$

For large N,

$$I_{2(N-1)}(x) \sim I_{2N}(x) \tag{35}$$

as $I_N(x)$ is just the volume of an N-dimensional sphere of radius x. Let us define

$$\begin{aligned} A &= \frac{p_1^2}{2mk_BT_\alpha} \\ B &= \frac{1}{2\pi mk_BT_\alpha}. \end{aligned} \tag{36}$$

With these, the terms for the normalization factor can be rewritten as:

$$\mathrm{O}(1) : \left(\frac{L}{h}\right)^{2N} 2I_{2N}(2mE_\alpha) \sum_{m=0}^{\infty} \sum_{k=-2m,-2m+2,\ldots 2m} \cos(\pi k\nu)\mathcal{P}_k^M(N) \tag{37}$$

and

$$\mathrm{O}\left(\frac{h^2}{L^2}\right) : \left(\frac{L}{h}\right)^{2(N-1)} I_{2N}(2mE_\alpha)B \times \sum_{m=0}^{\infty} \sum_{k=-2m-1,-2m+1,\ldots 2m+1} [\cos(\pi k\nu)\mathcal{R}_k^M(N) + \cos(\pi k\nu)\mathcal{Q}_k^M(N)]. \tag{38}$$

Similarly, the terms corresponding to the numerator of (19) can be rewritten as

$$\mathrm{O}(1) : \left(\frac{L}{h}\right)^{2N} 2I_{2N}(2mE_\alpha)B\exp(-A) \sum_{m=0}^{\infty} \sum_{k=-2m,-2m+2,\ldots 2m} \cos(\pi k\nu)\mathcal{P}_k^M(N) \tag{39}$$

and

$$\mathrm{O}\left(\frac{h^2}{L^2}\right) : \left(\frac{L}{h}\right)^{2(N-1)} I_{2N}(2mE_\alpha)B^2\exp(-A) \times \sum_{m=0}^{\infty} \sum_{k=-2m-1,-2m+1,\ldots 2m+1} [\cos(\pi k\nu)\mathcal{R}_k^M(N) + 2\cos(\pi k\nu)\mathcal{Q}_k^M(N)]. \tag{40}$$

Equations (41) and (42) combine to give the numerator of (19) which we call $\mathcal{F}_1$, and, equations (39) and (40) combine to give the denominator of (19) which we call $\mathcal{F}_2$. Thus the ratio of $\mathcal{F}_1$ to $\mathcal{F}_2$ gives the momentum distribution up to $\mathrm{O}(h^2/L^2)$. Now, after a straightforward arrangement of all the terms, we get

$$F(\boldsymbol{p}_1) = (2\pi mkT)^{-1}\exp\left(-\frac{\boldsymbol{p}_1^2}{2mkT}\right) \times \left\{1 + \left(\frac{h}{L}\right)^2 \frac{1}{2\pi mkT}(2e^{-\frac{p_1^2}{2mkT}} - 1)G(N,\nu) + \mathrm{O}\left(\frac{h^4}{L^4}\right)\right\} \tag{41}$$

where

$$G(N,\nu) = \frac{\sum_{m=0}^{\infty}\sum_{K=-2m-1(\text{even})}^{2m+1} Q_K^{(m)}(N)\cos(\pi K\nu)}{1 + 2\sum_{m=1}^{\infty}\sum_{K=-2m(\text{odd})}^{2m} P_K^{(m)}(N)\cos(\pi K\nu)} \tag{42}$$

$Q_K^{(m)}$ is the number of elements in B_N composed of 'm' generators whereby the momentum $\boldsymbol{p}_1$ is interchanged with another momentum yielding a character $\exp(\mathrm{i}\pi K\nu)$ (or $\exp(-\mathrm{i}\pi K\nu)$ since $Q_K^{(m)}(N) = Q_{-K}^{(m)}(N)$); $P_K^{(m)}(N)$ is the number of elements in B_N contributing to identity with a character $\exp(\mathrm{i}\pi K\nu)$ (or $\exp(-\mathrm{i}\pi K\nu)$). Temperature is introduced above via the ideal gas law, $E_\alpha = NkT_\alpha$. Unfortunately though, this counting problem stands open today [27]. It is very important to note that the ansatz (11)–(13) for the special case when $\sigma_i^2 = 1$ for all i where B_N reduces to symmetric group, S_N, the well known Fermi–Dirac and Bose–Einstein distributions follow. In view of evaluating pressure, Π from (14), denoting the area of the enclosure by A, we get $\Pi A/kT = 1 - (2A)^{-1}\lambda^2 G(N, \nu)$, with $\lambda^2 = h^2(2\pi mkT)^{-1}$. We immediately see that $G(N, 0)/(2N)$ and $G(N, 1)/(2N)$ are $2^{-3/2}$ and $-2^{-3/2}$ respectively yielding the second virial coefficient for the Bose and Fermi gases [28]. For the fractional case, with $\nu =$ even number, $2j + \delta$ ('boson-based anyons'), comparing our result with [14], we get the sum rule:

$$N^{-1}G(N, \nu) = -1 + 4|\delta| - 2\delta^2 \tag{43}$$

the right-hand side belongs to [14]. It is important to note that our deduction is non-perturbative and in principle, we can get expressions for higher-order virial coefficients also [29]. To understand this, we observe that relation (14) connects two momentum configurations of N particles, and not just the momenta of two particles. Thus, it contains information that can lead to all virial coefficients. For example, for the third virial coefficient, we need to evaluate contributions to $F(\boldsymbol{p}_1)$ when three momenta out of N are interchanged. The denominator of (19) contains those interchanges which braid three strands in such a way that the initial configuration of momenta is preserved whereas the numerator of (19) contains those which exchange the momentum assignment on all three strands. We have done the calculation and the third virial coefficient is expressible in terms of the specific counting problem of B_N. Here, in order to convince the reader, it suffices to make a comparative discussion with the existing calculation. For this, we write down the total contribution to the momentum distribution due to a triple interchange emerging from the elements of B_N formed by M generators,

$$\left(\frac{L}{\hbar}\right)^{2(N-2)} \sum_{-2M}^{2M} \frac{2}{3} I_{2(N-3)}(2mE_\alpha - \boldsymbol{p}_1^2)\cos(\pi k\nu)\mathcal{R}_k^M(N) + 2I_{2(N-3)}(2mE_\alpha - 3\boldsymbol{p}_1^2)\cos(\pi k\nu)\mathcal{S}_k^M(N) \tag{44}$$

where $\mathcal{S}_k^M(N)$ ($\mathcal{R}_k^M(N)$) are the number of elements of B_N that (do not) change the momentum $\boldsymbol{p}_1$. $I_D(x)$ denotes the volume of a D-dimensional hyper-sphere of radius x. The reason we give this result here is to show that (46) is a Fourier series with harmonic terms such as $\cos 2\pi\nu, \cos 4\pi\nu$, etc, in agreement with the conjectured form [29]. It is becoming evident from recent calculations [30] that the third virial coefficient is a series with terms such as $\sin^2 \pi\nu, \sin^4 \pi\nu$, etc. Our formal result is thus in consonance with these works.

4. Average entropy of a quantum subsystem—averaging over random eigenstates, and, over random Hamiltonians

In this section, our discussion will not be restricted to two dimensions as we now come to the thermodynamics of the different mathematical schemes we have been discussing above. We have seen above that guided by RMT, and extending the treatment in [9], we obtain a formal expression for the momentum distribution of an ideal anyon gas. In our treatment, there is no thermal bath, and to see if thermalization actually occurs, we calculate entropy. We will show that an assumption of 'an eigenstate picked randomly for an isolated system' or 'a system being in a random matrix ensemble' maximizes entropy in consistency with the second law of thermodynamics.

For this purpose, we consider a quantum version of the Ehrenfest urn model [31]. We are fortunate that a treatment of this problem [22] already exists, applied lately in the context of black hole evaporation problems [32].

Consider an isolated system AB with Hilbert space dimension mn and normalized density matrix $\hat{\rho}$ (a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ if $\hat{\rho}^2 = \hat{\rho}$) [33, 32]. Now we divide this system into two subsystems, A and B, of dimensions m and n respectively. The density matrices of A and B, respectively, $\hat{\rho_A}$ and $\hat{\rho_B}$, are obtained by partial tracing of $\hat{\rho}$ over B and A respectively. We assume that A and B are quantally uncorrelated, i.e. $\hat{\rho} = \hat{\rho_A} \otimes \hat{\rho_B}$. If $|\psi\rangle$ is chosen at random, what is the joint probability distribution of eigenvalues of $\hat{\rho_A}$? Following [22], 'random' refers to unitarily invariant Haar measure which, in this case, turns out to be hyper-area of the unit sphere $\mathcal{S}^{2mn-1}$, the factor of 2 coming from the fact that $|\psi\rangle$ has mn complex entries (or $2mn$ real entries). The objective is to study the average entropy of A, $\langle S_A\rangle (= -\,\mathrm{tr}\,\hat{\rho_A}\log\hat{\rho_A})$ over the probability distribution of eigenvalues (which are probabilities) of $\hat{\rho_A}$. The result of this calculation, conjectured in [32] and proved first in [34], is

$$\langle S_A\rangle = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \qquad (m \leqslant n). \tag{45}$$

The random pure state can be written (for the system we are considering) as (7), (8) in three (or greater) space dimensions, or, as (13)–(15) in two dimensions. It is an important conceptual point to note that these equations are a mathematical representation of choosing a pure state at random for a specific choice of amplitude, A_α.

If one calculates the average of trace of $\hat{\rho_A}^2$, first, over homogeneously distributed unit vector in mn-dimensional Hilbert space, and then, over random Hamiltonians, the two answers are only different by one bit [35]. The values are almost the same as one corresponding to the answer when the entropy will be maximal, i.e. when each probability is $1/m$ and hence the entropy is $-\sum_{i=1}^{m}\frac{1}{m}\log\frac{1}{m} = \log m$. This brings us to random matrices as all that is being done here about averaging over random Hamiltonians is what is done in RMT. Thus the average entropy of a subsystem follows from the random matrix hypothesis about AB—the statement becomes exact when $m \ll n$. Indeed, the connection of RMT and statistical mechanics is when the size of the system is large where it means then that the number of particles is large to be consistent with thermodynamic limit.

In fact, it has been shown [36] that if the pure states of AB are random, the probability distribution of eigenvalues of $\hat{\rho_A}$ is just the one-point correlation function corresponding to the (generalized) Laguerre unitary ensemble of random matrices [38]. Since the one-point correlation function (average level density) is the same for orthogonal, unitary, and symplectic ensembles [37], the answer for the average entropy will remain the same. Let us remember that the average level density is, in principle, a function of the size of the matrices. The fact that we must discuss entropy in the context of systems in the thermodynamic limit is what makes the entropy the same for all the ensembles. Since the fluctuations on top of the average density become significant with decreasing sizes, we expect to observe their interesting effect on the entropy. We now present results that prove these remarks.

We begin by recalling that the average over all pure states of AB, in unitary Haar measure, of the spread of eigenvalues of $\hat{\rho_A}$ is [22]

$$\langle\sigma^2\rangle = \left\langle \frac{1}{m}\sum_{i=1}^{m}\left(p_i - \frac{1}{m}\right)^2 \right\rangle = \frac{1-m^{-2}}{mn+1}. \tag{46}$$

Clearly, the case $n = 1$ corresponds to the situation when $\hat{\rho_A}$ is also pure, then we have

$$\langle\sigma^2_{\max}\rangle = \frac{1-m^{-2}}{m+1} \tag{47}$$

the ratio of $\langle\sigma^2\rangle$ to $\langle\sigma^2_{\max}\rangle$ gives the measure of 'purity' of the subsystem A. Since we have noted above that the same answer can be obtained by averaging over the one-point correlation function of the Laguerre unitary ensemble of random matrices, we expect that apart from the leading term which is $\log m$ for the entropy as this corresponds to equipartition, the 'defect' term must show the signature of different ensembles. To this end, we start by writing the entropy,

$$S = -\sum_i p_i \log p_i \tag{48}$$

and Taylor expand each p_i about $1/m$. With

$$p_i = \frac{1}{m} - \frac{q_i}{m} \tag{49}$$

we can write

$$S = \log m - \frac{1}{m}\left[\frac{1}{1.2}\sum_{i=1}^{m} q_i^2 + \frac{1}{2.3}\sum_{i=1}^{m} q_i^3 + \cdots\right] \tag{50}$$

which is convergent if $|q_i| < 1$, i.e. if $0 < p_i < \frac{2}{m}$. Since $\langle\sigma^2\rangle$ is small for large n, most of the measure will lie with $p_i < \frac{2}{m}$. It is plausible that $S = \log m$—defect, and that the defect is well approximated by

$$\begin{aligned}\langle \text{ defect } \rangle \equiv \frac{1}{2m}\sum_i q_i^2 &= \frac{1}{2}m^2\sigma^2 \\ &= \frac{1}{2}\frac{m^2-1}{mn+1}.\end{aligned} \tag{51}$$

We can now find the defect for the case where we integrate over orthogonally invariant Haar measure [40] and symplectically invariant Haar measure. The difference is that these correspond to hyper-spheres of dimensions $(mn-1)$, $\mathcal{S}^{mn-1}$, and $(4mn-1)$, $\mathcal{S}^{4mn-1}$ respectively. After the same steps, we get

$$\left\langle\frac{\sigma^2}{\sigma^2_{\max}}\right\rangle = \frac{\frac{\beta}{2}m+1}{\frac{\beta}{2}mn+1} \tag{52}$$

where β is the codimension of level crossing, respectively 1, 2, and 4 for orthogonal, unitary, and symplectic ensembles of RMT. The entropy is given by

$$S = \log m - \frac{(m-1)}{2}\frac{\frac{\beta}{2}m+1}{\frac{\beta}{2}mn+1} + \mathrm{O}(n^{-2}). \tag{53}$$

This result shows that the entropy is almost maximal for n large enough, and that the finite-dimensional effects contain, as expected, the dependence on the global symmetries of the system. We also see that for the purpose of average entropy calculation, chirality does not play an explicit role. This observation is also intuitively expected from the relation between thermodynamics and statistical mechanics.

We wish to note that the arguments we have used to reach the second law of thermodynamics are quantum mechanical. The results of this section show that randomness in eigenstate, which follows from RMT, encompassing the ergodicity of the classical system, leads to entropy of a subsystem which is maximal. To notice the differences between the chaotic quantum systems belonging to different random matrix ensembles, we need to study the 'defect' in entropy as a function of the Hilbert space dimension as given in the final formula above. We conclude this section with the mention of a recent work where a related study is carried out on periodically kicked top [18].

5. Concluding remarks

In this paper we have shown that an ansatz where eigenstates are written as random superpositions of plane waves (or some basis consistent with the boundary conditions) for systems whose classical analogues are chaotic is equivalent to a random matrix hypothesis. In an important work [9], the connection between the ansatz and statistical mechanics in three dimensions was brought out. Since the group that governs exchange symmetry in two dimensions is an infinite, non-Abelian one whose special case is the permutation group (the exchange group in three dimensions), our treatment and results in section 3 are generalizations of [9]. It was possible for us to do this only because we realized what the random matrix ensemble should be in two dimensions. This is the reason for section 2 where a comparative discussion about the relevance of space dimensions is given to the random matrix ensembles. In consistency with the expectations, the ensemble in two dimensions is chiral unitary ensemble. It is well known that choosing a specific nature of randomness (e.g. Gaussian) then gives the average density of states which is not realistic. This can be treated with the Dyson Brownian motion model [20] where any realistic density of states can be modelled. An interesting relation between a generalized Brownian motion model and a semiclassical reasoning of universality has recently been worked out [39] by formulating a hydrodynamic description.

The fact that both the ideas, one of random pure state of an isolated system, and, that of this system being governed by a random Hamiltonian, give rise to the distribution functions and virial coefficients correctly suggest that there may be a connection between the two. In section 4, we described a way that we see most clearly (as of now) in the context of entropy of a quantum subsystem. We believe that the arguments developed here provide a common ground to seemingly different themes, leading to the second law of thermodynamics. In conclusion, in this paper, we have brought together quantum chaos, RMT, and statistical mechanics for the special case of ideal gases. A more general conclusion will be highly desirable.

As shown in section 3, the number of words formed by M generators of the Braid group satisfy a sum rule which comes from our calculation of the second virial coefficient. Currently, we are involved in understanding the relevant counting problem explicitly and hope to give an answer in the future.

Finally, the parameter, ν has an analogous partner in quantum chromodynamics [14] and we conjecture that the anyon gas discussed here and the $\nu\pi$-parametrized quantum chromodynamics belong to the same universality class of chiral unitary ensemble of RMT.

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